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Local Accuracy for Radial Basis Function Interpolation on Finite Uniform Grids

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We consider interpolation on a finite uniform grid by means of one of the radial basis functions (RBF) $\phi(r) = r^\gamma$ for $\gamma > 0$, $\gamma \notin 2\mathbb{N}$ or $\phi(r) = r^\gamma \ln r$ for $\gamma \in 2\mathbb{N}_+$. For each positive integer N, let $h = N^{-1}$ and let $\{x_i : i = 1, 2, ..., (N+1)^d\}$ be the set of vertices of the uniform grid of mesh-size h on the unit d-dimensional cube $[0, 1]^d$. Given $f: [0, 1]^d \to \mathbb{R}$, let s_h be its unique RBF interpolant at the grid vertices: $s_h(x_i) = f(x_i)$, $i = 1, 2, ..., (N+1)^d$. For $h \to 0$, we show that the uniform norm of the error $f - s_h$ on a compact subset K of the interior of $[0, 1]^d$ enjoys the same rate of convergence to zero as the error of RBF interpolation on the infinite uniform grid $h\mathbb{Z}^d$, provided that f is a data function whose partial derivatives in the interior of $[0, 1]^d$ up to a certain order can be extended to Lipschitz functions on $[0, 1]^d$.

Key Words: radial basis function interpolation; local error estimates; finite uniform grids.

1. INTRODUCTION

Motivation. Several authors (Jackson [7], Powell [13], Schaback [16]) have expressed an interest in the accuracy of RBF interpolation on finite uniform grids. The question suggested by Powell's work [13] is whether these results would reproduce the high orders of accuracy that occur on infinite uniform grids not only for the case of the multiquadric and linear radial functions, but also for other radial basis functions that are currently studied. We provide a positive answer to this question in Theorem 1. In addition, for a specific class of radial basis functions, Theorem 2 shows that one cannot expect higher local rates of accuracy for general smooth data functions.

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Notation and Problem Description. Let d and N be positive integers and denote $h=N^{-1}$, $D=[0,1]^d$, $\mathcal{J}_N=\mathbb{Z}^d\cap h^{-1}D$, $\mathcal{V}_h=h\mathcal{J}_N$. Hence $\mathcal{V}_h$ contains all the vertices of the uniform grid of mesh size h on the unit d-dimensional cube D. Let $f\colon D\to\mathbb{R}$ be a data function whose partial derivatives in the interior of D up to a certain order can be extended to Lipschitz functions on D.

For sufficiently large N, we consider interpolation to f on $\mathcal{V}_h$ by means of one of the radial basis functions $\phi(r)=r^\gamma$ for $\gamma>0$, $\gamma\notin 2\mathbb{N}$ and $\phi(r)=r^\gamma\ln r$ for $\gamma\in 2\mathbb{N}_+$. In any of these cases, let m be the integer part of $\gamma/2$ and note that the $(m+1)$-st derivative of the function $\Phi\colon[0,\infty)\to\mathbb{R}$, $\Phi(r)=\phi(r^{1/2})$ is strictly completely monotonic, i.e., $(-1)^k\,\Phi^{(k+m+1)}(r)>0$ for $r>0$ and $k\in\mathbb{N}$. Therefore, by the classical theory of Micchelli (cf. [12, 13]), there exists a unique function s_h of the form

$$s_h(x)=\sum_{j\in\mathcal{J}_N} c_j\phi(\|x-hj\|)+p_h(x),\qquad x\in\mathbb{R}^d, \tag{1.1}$$

that satisfies the conditions

$$s_h(hj)=f(hj),\qquad j\in\mathcal{J}_N, \tag{1.2}$$

$$\sum_{j\in\mathcal{J}_N} c_j p(hj)=0,\qquad \forall p\in\Pi_m(\mathbb{R}^d), \tag{1.3}$$

where p_h belongs to $\Pi_m(\mathbb{R}^d)$—the space of polynomials on $\mathbb{R}^d$ of total degree not exceeding m—and $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^d$.

The problem considered in this paper is to investigate the rate of convergence of $\|f-s_h\|_{L^\infty(K)}$ to zero as $h\to 0$, where K is any fixed compact subset of the interior of D.

Method of Analysis. A similar problem has been considered by Powell in [13, Sect. 9]. Namely, that work deals with the case when ϕ is the multiquadric or linear radial function, d is odd, the radial translates $\phi(\|x-hj\|)$ in (1.1) are replaced by $\phi(\|h^{-1}x-j\|)$, while $p_h\equiv 0$ (i.e., the polynomial term of s_h is removed and (1.3) is omitted). Powell's method relates the error of interpolation on $\mathcal{V}_h$ to the error of interpolation on the infinite grid $h\mathbb{Z}^d$ for a suitable extension f^* of f to $\mathbb{R}^d$. Specifically, for any fixed x in the interior of D, we have

$$f(x)-s_h(x)=I_hf^*(x)-s_h(x)+f^*(x)-I_hf^*(x), \tag{1.4}$$

where I_hf^* is the RBF interpolant to f^* on $h\mathbb{Z}^d$ (see next subsection). By a careful analysis of the difference $I_hf^*(x)-s_h(x)$ in the right-hand side of (1.4) and by making use of the well known results concerning the error $f^*(x)-I_hf^*(x)$, Powell deduces that the left-hand side of (1.4) enjoys the same rate of convergence to zero in the interior of D, as the error of interpolation on the infinite grid $h\mathbb{Z}^d$. We present a modification of this method that gives the same conclusion for the problem formulated in the previous subsection.

RBF Interpolation on the Infinite Uniform Grid. The two types of basis functions considered in this paper are treated as Examples 5–1 and 5–3 in the fundamental work of Buhmann [3] (see also [4]). He proves the existence of a unique cardinal function

$$\chi(x) = \sum_{j \in \mathbb{Z}^d} \mu_j \phi(\|x-j\|), \qquad x \in \mathbb{R}^d, \tag{1.5}$$

where μ_j, $j \in \mathbb{Z}^d$, are real coefficients such that χ is defined by an absolutely convergent series and satisfies $\chi(j) = \delta_{0j}$ for all $j \in \mathbb{Z}^d$. Moreover, there exists a maximal integer $\kappa \geqslant 1$ such that the interpolant

$$Ip(x) = \sum_{j \in \mathbb{Z}^d} p(j)\, \chi(x-j), \qquad x \in \mathbb{R}^d, \tag{1.6}$$

is well defined and $Ip = p$ for all $p \in \Pi_\kappa(\mathbb{R}^d)$ (i.e., interpolation on $\mathbb{Z}^d$ reproduces polynomials of degree up to κ). Specifically, for any ϕ as above, κ is the least integer greater than or equal to $d+\gamma-1$.

We will also need the rate of decay of χ for large x. For both types of RBFs ϕ and for any dimension d, the bound

$$|\chi(x)| \leqslant c\, \|x\|^{-l}, \qquad \forall x \neq 0, \tag{1.7}$$

is satisfied with $l = 2d + \gamma$, where c denotes a generic constant. In addition, if d is odd and $\phi(r) = r^\gamma$, $\gamma \in 2\mathbb{N}+1$, or if d is even and $\phi(r) = r^\gamma \ln r$, $\gamma \in 2\mathbb{N}_+$, then χ enjoys an exponential rate of decay at infinity (cf. Madych and Nelson [10]).

A consequence of these results is that, for functions $f^*\colon \mathbb{R}^d \to \mathbb{R}$ that have continuous partial derivatives up to order $\kappa+1$, with Lipschitz and/or boundedness assumptions on some of them, the formula

$$I_h f^*(x) = \sum_{j \in \mathbb{Z}^d} f^*(hj)\, \chi(h^{-1}x - j), \qquad x \in \mathbb{R}^d, \tag{1.8}$$

defines an interpolant to f^* on the scaled infinite grid $h\mathbb{Z}^d$. Further, in addition to the interpolation conditions

$$I_h f^*(hj) = f^*(hj), \qquad j \in \mathbb{Z}^d, \tag{1.9}$$

$I_h f^*$ has the property

$$\|f^* - I_h f^*\|_{L^\infty(\mathbb{R}^d)} = \mathcal{O}(h^{d+\gamma}), \qquad \text{for} \quad h \to 0. \tag{1.10}$$

The arguments that prove (1.10) are discussed in Section 2. We remark that the above accuracy orders are known to be sharp, in the sense that there exists a smooth function f^* for which $\|f^* - I_h f^*\|_{L^\infty(\mathbb{R}^d)}$ does not tend to zero faster than $\mathcal{O}(h^{d+\gamma})$ as $h \to 0$ (cf. de Boor and Ron [2], Buhmann [3], Jackson [6]).

The Difference $I_h f^* - s_h$. An important step in Powell's method of analysis described above is provided by the identity

$$
\begin{aligned}
I_h f^*(x) - s_h(x) & \\
&= I_h(f^* - s_h)(x) \\
&= \sum_{j \in \mathbb{Z}^d} [f^*(hj) - s_h(hj)]\, \chi(h^{-1}x - j), \qquad x \in D, \qquad (1.11)
\end{aligned}
$$

which will be justified in the next section. Note that, by (1.2) and the fact that f^* is an extension of f, only the indices $j \notin \mathcal{J}_N$ give a nonzero term on the right-hand side of (1.11). Further, f^* will be chosen in such a way that $\operatorname{supp}(f^*)$ is compact, so $|f^*(hj) - s_h(hj)| = |s_h(hj)|$ is satisfied for all but a finite number of indices $j \in \mathbb{Z}^d$. Therefore, in order to exploit the asymptotic decay (1.7) of the cardinal function χ in (1.11), we need an estimate of the growth of $|s_h(x)|$ for $\|x\| \to \infty$, independent of h. Specifically, by using the variational approach of Wu and Schaback [21], we prove in Proposition 1 of the next section that, whenever ϕ is one of the radial functions considered in this paper, there exists $\tau > 0$ with the property

$$
|s_h(x)| \leqslant c_f (1 + \|x\|)^\tau, \qquad x \in \mathbb{R}^d, \qquad (1.12)
$$

for all sufficiently small h, where $c_f > 0$ depends on f, but not on h.

The Results. A bound of the type (1.12) and the relations between the numbers κ, l, τ are the key ingredients that allow the extension of Powell's approach to the present case.

Before stating the main result, we introduce the Lipschitz spaces $\mathrm{Lip}(k+1, F)$, where k is a nonnegative integer and $F \subset \mathbb{R}^d$ is a closed set (see Stein [20, p. 176] for the general definition). When $F \subset \mathbb{R}^d$ is a closed nonempty set such that $F = \mathrm{cl}(\mathrm{int}(F))$, these spaces are characterized as follows. $\mathrm{Lip}(1, F)$ is the space of bounded and Lipschitz continuous functions on F. For $k \geqslant 1$, $\mathrm{Lip}(k+1, F)$ is the space of bounded continuous functions f on F with the properties: f is of class C^k on the interior of F, its partial derivatives $\partial^\alpha f / \partial x^\alpha$ extend to bounded continuous functions on F for all multi-indices α such that $|\alpha| \leqslant k$, and the extended functions $\partial^\alpha f / \partial x^\alpha$ on F belong to $\mathrm{Lip}(1, F)$ when $|\alpha| = k$.

The reason for working with Lipschitz spaces is that we use the Whitney theorem to extend the given data function $f\colon D \to \mathbb{R}$ to $f^*\colon \mathbb{R}^d \to \mathbb{R}$ and we need some of the partial derivatives of f^* to be Lipschitz. Note also that the boundedness requirement for the partial derivatives of f in the above definition is superfluous when $F = D$.

THEOREM 1. *Let ϕ be any one of the basis functions mentioned in the second subsection and let $f \in \mathrm{Lip}(\kappa+2, D)$. Then for any compact subset K of the interior of the unit d-cube D, the error of the interpolant* (1.1) *is bounded by the inequality*

$$\|f - s_h\|_{L^\infty(K)} \leqslant ch^{d+\gamma}, \qquad \text{as} \quad h \to 0, \tag{1.13}$$

where c is a positive constant that depends on f and K, but not on h.

The remark after Eq. (1.10) and numerical experiments suggest that the rate (1.13) is sharp. The following result proves this suggestion for the case when d and ϕ are such that the cardinal function (1.5) has an exponential rate of decay at infinity. A proof for the remaining cases is unknown at present.

THEOREM 2. *If d is odd and $\phi(r) = r^\gamma$, $\gamma \in 2\mathbb{N}+1$ or if d is even and $\phi(r) = r^\gamma \ln r$, $\gamma \in 2\mathbb{N}_+$, then the order of convergence* (1.13) *is sharp, in the sense that there exist a function $f \in \mathrm{Lip}(\kappa+3, D)$ and a compact set $K \subset \mathrm{int}(D)$ for which the left-hand side of* (1.13) *does not tend to zero faster than $\mathcal{O}(h^{d+\gamma})$, as $h \to 0$.*

The proofs of Theorems 1 and 2 are given in Sections 2 and 3, respectively. They also apply, with minor modifications, to the case when the unit cube D is replaced by any set that is the closure of an open bounded domain in $\mathbb{R}^d$.

Comments. This work was initiated by an interest in the basis function ϕ: $[0, \infty) \to \mathbb{R}$, $\phi(r) = r^2 \ln r$, which gives the *thin plate spline* (TPS) interpolant when $d = 2$. In this case, our results imply the rate of convergence $\mathcal{O}(h^4)$ for TPS interpolation inside the unit square.

For the corresponding one dimensional interpolant using the same basis function, Theorem 1 gives $\mathcal{O}(h^3)$ accuracy on compact sets inside $[0, 1]$. This conclusion is confirmed by numerical experiments performed by the author, which also indicate a deterioration in the order of accuracy to $\mathcal{O}(h^{3/2})$ near the endpoints of the interval, for general smooth data functions. Hence, if we want the uniform error behavior on $[0, 1]$ to be closer to the local one for smooth data functions, a modification of the interpolation method is needed. Indeed, if we replace the (two) constraints (1.3) with the condition that s_h interpolates the end derivatives, numerical experiments suggest that the convergence rate is improved to $\mathcal{O}(h^{5/2})$ near the ends of the interval. These features are under current investigation.

Note that, for $d = 1$ and $\phi(r) = r^{2m+1}$, $m \in \mathbb{N}$, the interpolant defined by (1.1)–(1.3) is a *natural* spline of degree $2m+1$. Our results give the rate of convergence $\mathcal{O}(h^{2m+2})$ in the interior of an interval, which is well known (e.g., Atkinson [1]).

Literature. The accuracy of interpolation at a *finite* number of (possibly scattered) points inside D by means of the radial basis functions considered in this article has been the subject of much research, as illustrated by the papers of Duchon [5], Wu and Schaback [21], Powell [14], Schaback [17, 18], Johnson [8, 9], for example. Using different techniques, these papers obtain results for the *uniform* norm of the error on D, where D may not be the unit cube. Therefore, due to boundary effects, it is usual for their uniform rates of convergence to be slower than the maximal *local* order $\mathcal{O}(h^{d+\gamma})$ of Theorem 1, which is the main subject of our work. Similar results to Theorem 1 have been also obtained by Matveev [11] for the case of scattered interpolation points, but his methods only apply to the special cases that form the hypothesis of Theorem 2. By contrast, our proof of Theorem 1 covers the entire ranges of γ and d.

2. PROOF OF THEOREM 1

The Extension f^* of f. The method of proof that is described in the Introduction requires the construction of a suitable extension f^* of f to $\mathbb{R}^d$, which is done in two steps. Firstly, since $f \in \mathrm{Lip}(\kappa+2, D)$, by the Whitney extension theorem (cf. Stein [20, Chap. VI]), there exists $\tilde{f} \in \mathrm{Lip}(\kappa+2, \mathbb{R}^d)$ such that $\tilde{f}(x) = f(x)$, $x \in D$. Secondly, we let $\eta \in C_0^\infty(\mathbb{R}^d)$ be a cut-off function with $\eta(x) = 1$, $x \in D$, and $\eta(x) = 0$ for sufficiently large $\|x\|$. Hence $f^* = \eta \tilde{f} \in \mathrm{Lip}(\kappa+2, \mathbb{R}^d)$ satisfies

$$f^*(x) = f(x), \qquad x \in D, \tag{2.1}$$

and $\mathrm{supp}(f^*)$ is compact. Note that the extension f^* is not unique.

The Order of Convergence (1.10). We recall that κ is the least integer that satisfies $\kappa \geqslant d + \gamma - 1$. If $f^* \in C^{\kappa+1}(\mathbb{R}^d)$ and f^* has bounded κth and $(\kappa+1)$st-order partial derivatives, the property (1.10) of the error of interpolation to f^* on $h\mathbb{Z}^d$ is proved by Buhmann [3] for all the considered choices of ϕ with the exception of the case when γ is a positive integer and κ is even (see also [4]). In this special case, Buhmann's results imply only that the interpolation error is at most a constant multiple of $h^{d+\gamma}\,|\ln h|$ as $h \to 0$. However, our conditions on f^* ensure that inequality (1.10) holds for this case too, as shown by the cancellation argument of Powell [13, Proof of Theorem 8.5]. Indeed, since $f^* \in \mathrm{Lip}(\kappa+2, \mathbb{R}^d)$, it follows that f^* satisfies in particular not only Buhmann's assumptions, but also the hypotheses of [13, Theorem 8.5]. That theorem also requires a condition on ϕ. Specifically, for all the choices of ϕ considered by our paper, the distributional Fourier transform of ψ: $\mathbb{R}^d \to \mathbb{R}$, $\psi(x) = \phi(\|x\|)$, is given by $\hat{\psi}(t) = c\, \|t\|^{-d-\gamma}$ for $t \neq 0$. Therefore equation (8.34) from [13] is true when $\gamma \in \mathbb{N}_+$ and κ is even. Hence Lemma 8.4 of [13] also holds true, and then the proof of Theorem 8.5 in [13] gives the order of convergence (1.10).

The Justification of (1.11). By (1.8), it suffices to prove the identity

$$s_h(x) = \sum_{j \in \mathbb{Z}^d} s_h(hj)\, \chi(h^{-1}x - j), \qquad x \in \mathbb{R}^d, \tag{2.2}$$

or, after a change of variables,

$$s_h(hx) = \sum_{j \in \mathbb{Z}^d} s_h(hj)\, \chi(x - j), \qquad x \in \mathbb{R}^d. \tag{2.3}$$

Letting $p = p_h(h\,\cdot)$ in (1.6), where p_h occurs in the definition (1.1), and noting that $m < \kappa$, the polynomial reproduction property shows that (2.3) is equivalent to

$$s_h(hx) - p_h(hx) = \sum_{j \in \mathbb{Z}^d} [s_h(hj) - p_h(hj)]\, \chi(x - j), \qquad x \in \mathbb{R}^d. \tag{2.4}$$

By (1.1), the left-hand side of (2.4) is the function

$$\sum_{j \in \mathscr{I}_N} c_j \phi(h\, \|x - j\|), \qquad x \in \mathbb{R}^d. \tag{2.5}$$

When $\phi(r) = r^\gamma$, $\gamma > 0$, $\gamma \notin 2\mathbb{N}$, homogeneity implies that (2.5) is a linear combination of terms of the form $\phi(\|x - j\|)$ for $j \in \mathscr{I}_N$. Alternatively, when $\phi(r) = r^\gamma \ln r$, $\gamma \in 2\mathbb{N}_+$, (2.5) becomes

$$\begin{aligned}
&\sum_{j \in \mathscr{I}_N} c_j \phi(h\, \|x - j\|) \\
&\qquad = \sum_{j \in \mathscr{I}_N} c_j h^\gamma\, \|x - j\|^\gamma\, (\ln h + \ln \|x - j\|) \\
&\qquad = h^\gamma \sum_{j \in \mathscr{I}_N} c_j \phi(\|x - j\|) + h^\gamma \ln h \sum_{j \in \mathscr{I}_N} c_j\, \|x - j\|^\gamma,
\end{aligned} \tag{2.6}$$

and the last sum is a polynomial of degree less than γ (cf. (1.3)), which is reproduced by interpolation on $\mathbb{Z}^d$, since $\kappa \geqslant d+\gamma-1 \geqslant \gamma$. Hence the required equation (2.4) is satisfied if interpolation on the infinite grid $\mathbb{Z}^d$ reproduces the function $\phi(\|\cdot - j\|)$, for every $j \in \mathscr{J}_N$.

We prove this condition for the above choices of ϕ by applying arguments from [13, Sect. 7]. Specifically, we consider the linear space $\mathscr{S}$ defined by (7.16), (7.17) of that paper. Due to Theorem 5-7 of [3], the cardinal function (1.5) is in $\mathscr{S}$. Further, it is elementary that any choice of ϕ from Section 1 satisfies inequality (7.18) of [13]. It follows that both the arguments of Lemma 7.5 and of page 169 of [13] are valid for our choices of ϕ. Thus interpolation on the infinite grid $\mathbb{Z}^d$ reproduces the function $\phi(\|\cdot - k\|)$ for every $k \in \mathbb{Z}^d$, which completes the justification of (1.11).

The Uniform Growth of s_h.

PROPOSITION 1. *Let ϕ be one of the radial functions considered in this paper and assume that the data function f is in the set* $\operatorname{Lip}(\kappa+2, D)$. *Then the RBF interpolant* (1.1) *satisfies* (1.12), *where c_f is independent of h. Specifically, recalling that m denotes the integer part of $\gamma/2$, we have $\tau = \sigma + (m+\gamma)/2$, where $\sigma = 0$ for $\phi(r) = r^\gamma$, $\gamma > 0$, $\gamma \notin 2\mathbb{N}$, while σ can be any positive constant for $\phi(r) = r^\gamma \ln r$, $\gamma \in 2\mathbb{N}_+$.*

Proof. Let n be the dimension of the space $\Pi_m(\mathbb{R}^d)$ and denote by $\{P_j\colon j=1, 2, ..., n\}$ the monomial basis of $\Pi_m(\mathbb{R}^d)$. For any set $\{u_j\colon j=1, 2, ..., n\}$ of vectors from $\mathbb{R}^d$, we define

$$\omega(u_1, u_2, ..., u_n) = \det(P_j(u_i)). \tag{2.7}$$

Further, let $\mathscr{V} = \{v_j\colon j=1, 2, ..., n\}$ be a subset of D such that interpolation on $\mathscr{V}$ from the linear space $\Pi_m(\mathbb{R}^d)$ has a unique solution (for example, $\mathscr{V}$ may be the set of vertices of a suitable regular grid in any simplex that is included in D—cf. [15, p. 289]). Therefore we obtain the condition

$$\omega(v_1, v_2, ..., v_n) \neq 0, \tag{2.8}$$

and we define $\omega_0 = |\omega(v_1, v_2, ..., v_n)| > 0$. By the continuity of the function ω in all its arguments, there exists $\varepsilon > 0$ such that, for any subset $\{u_j\colon j=1, 2, ..., n\}$ of $\mathbb{R}^d$ whose elements satisfy $\|u_j - v_j\| < \varepsilon$, $j=1, 2, ..., n$, we have $|\omega(u_1, u_2, ..., u_n)| \geqslant \omega_0/2$. In particular, for sufficiently small h, there exists a subset $\{x_j\colon j=1, 2, ..., n\}$ of $\mathscr{V}_h$, such that

$$\|x_j - v_j\| < \varepsilon, \qquad j=1, 2, ..., n, \tag{2.9}$$

and

$$|\omega(x_1, x_2, ..., x_n)| \geqslant \frac{\omega_0}{2}. \tag{2.10}$$

To simplify the arguments later, denote the elements of the set $\mathscr{V}_h \backslash \{x_j : j = 1, 2, ..., n\}$ by $\{x_j : j = n+1, n+2, ..., (N+1)^d\}$. Moreover, we define the Lagrange interpolation polynomials $l_i \in \Pi_m(\mathbb{R}^d)$, $i = 1, 2, ..., n$, which satisfy $l_i(x_j) = \delta_{ij}$, $i, j = 1, 2, ..., n$, by the expression

$$l_i(x) = \frac{\omega(x_1, ..., x_{i-1}, x, x_{i+1}, ..., x_n)}{\omega(x_1, x_2, ..., x_n)},$$

$$x \in \mathbb{R}^d, \quad i = 1, 2, ..., n. \tag{2.11}$$

Consequently, we can write the Lagrange representation formula

$$p(x) = \sum_{j=1}^{n} l_j(x)\, p(x_j), \qquad p \in \Pi_m(\mathbb{R}^d), \quad x \in \mathbb{R}^d. \tag{2.12}$$

Next, using the extension f^* of f that is constructed at the beginning of this section, we let c_{f^*} be the positive number defined by

$$c_{f^*}^2 = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\widehat{f^*}(t)|^2 \, \|t\|^{d+\gamma} \, dt, \tag{2.13}$$

where $\widehat{f^*}$ is the classical Fourier transform of f^*. The finiteness of c_{f^*} can be deduced from the following properties of Fourier transforms

$$|t^\mu \widehat{f^*}(t)| = |\widehat{D^\mu f^*}(t)| \leqslant \|D^\mu f^*\|_{L^1(\mathbb{R}^d)}, \qquad t \in \mathbb{R}^d, \tag{2.14}$$

for a generic multi-index μ. The integrand of (2.13) is in $L^1(\mathbb{R}^d)$ if $|\widehat{f^*}(t)|^2 \, \|t\|^{d+\gamma} \leqslant c^2 \, \|t\|^{-d-\alpha}$ for some positive constants c, α and for large $\|t\|$, which amounts to

$$|\widehat{f^*}(t)| \, \|t\|^{d+(\gamma+\alpha)/2} \leqslant c. \tag{2.15}$$

By (2.14), this condition is true for some $\alpha > 0$, since $d + \gamma/2 < d + \gamma \leqslant \kappa + 1$, $f^* \in C^{\kappa+1}(\mathbb{R}^d)$ and $\operatorname{supp}(f^*)$ is compact.

Fix $x \in \mathbb{R}^d \backslash \mathscr{V}_h$. For $\beta_i \in \mathbb{R}$, $i = 1, 2, ..., (N+1)^d$, such that

$$p(x) = \sum_{i=1}^{(N+1)^d} \beta_i p(x_i), \qquad \forall p \in \Pi_m(\mathbb{R}^d), \tag{2.16}$$

denote

$$\mathscr{E}_x(\beta) = (-1)^{m+1} \left\{ \sum_{i=1}^{(N+1)^d} \sum_{j=1}^{(N+1)^d} \beta_i \beta_j \phi(\|x_i - x_j\|) - 2 \sum_{i=1}^{(N+1)^d} \beta_i \phi(\|x - x_i\|) \right\}. \tag{2.17}$$

The positivity of expression (2.17) is a consequence of the fact that $(-1)^{m+1}\phi(\sqrt{\cdot})$ is conditionally strictly positive definite of order $m+1$, for all our choices of ϕ. Let

$$\mathscr{P}(x) = \min\{\mathscr{E}_x(\beta)\}^{1/2}, \tag{2.18}$$

where the minimum is taken over all vectors β of $(N+1)^d$ components that satisfy (2.16). We also define $\mathscr{P}(x) = 0$ when $x \in \mathscr{V}_h$.

We now apply a result of Wu and Schaback [21, Theorem 4]. Since s_h interpolates f^* on $\mathscr{V}_h$ and c_{f^*} is finite, we obtain the pointwise bound

$$|s_h(x) - f^*(x)| \leqslant \mathscr{P}(x)\, c_{f^*}, \qquad x \in \mathbb{R}^d. \tag{2.19}$$

For each $x \in \mathbb{R}^d$, we choose $\beta_i = l_i(x)$, $i = 1, 2, ..., n$, and $\beta_i = 0$, $i = n+1, n+2, ..., (N+1)^d$, where l_i, $i = 1, 2, ..., n$ are the Lagrange polynomials (2.11). Hence (2.16) is satisfied due to (2.12), and the definition (2.18) implies

$$\mathscr{P}^2(x) \leqslant \sum_{i=1}^{n} \sum_{j=1}^{n} |l_i(x)|\, |l_j(x)|\, |\phi(\|x_i - x_j\|)| + 2 \sum_{i=1}^{n} |l_i(x)|\, |\phi(\|x - x_i\|)|. \tag{2.20}$$

The last part of the proof provides an estimate of the right-hand side of (2.20). We use (2.9)–(2.11), the continuity of the determinant function (2.7) in each of its variables, and the fact that each l_i, $i = 1, 2, ..., n$, is a polynomial of degree m over $\mathbb{R}^d$, to deduce the bound

$$|l_i(x)| \leqslant c(1 + \|x\|)^m, \qquad i = 1, 2, ..., n, \quad x \in \mathbb{R}^d, \tag{2.21}$$

where c is independent of x and h. Further, the numbers $|\phi(\|x_i - x_j\|)|$, for $i, j = 1, 2, ..., n$, are bounded by the constant $\max\{|\phi(\|x-y\|)|\colon x \in D, y \in D\}$, while $x_i \in D$, $i = 1, 2, ..., n$, and the growth of ϕ at infinity give

$$|\phi(\|x - x_i\|)| \leqslant c(1 + \|x\|)^{\gamma + 2\sigma}, \qquad x \in \mathbb{R}^d, \tag{2.22}$$

where c is still a generic constant. The last inequality holds with $\sigma = 0$ for $\phi(r) = r^\gamma$, $\gamma > 0$, $\gamma \notin 2\mathbb{N}$ and with any $\sigma > 0$ for $\phi(r) = r^\gamma \ln r$, $\gamma \in 2\mathbb{N}_+$. Consequently,

$$\mathscr{P}(x) \leqslant c(1 + \|x\|)^{\sigma + (m+\gamma)/2}, \qquad x \in \mathbb{R}^d, \tag{2.23}$$

for some updated constant c. Since f^* has compact support, relations (2.19) and (2.23) provide the required conclusion (1.12), with the value of τ that is specified in the statement of the proposition. ∎

Proof of (1.13). We now complete the proof of Theorem 1 by deducing a suitable bound on $I_h f^* - s_h$. Since our argument for sums of the type (1.11) is also needed twice in the proof of Theorem 2, it is synthesized in the next result.

PROPOSITION 2. *Let K_1 and K_2 be two compact subsets of $\mathbb{R}^d$, with $K_1 \subset \operatorname{int}(K_2)$, let χ: $\mathbb{R}^d \to \mathbb{R}$ be a function that satisfies the bound* (1.7) *for a positive exponent l, let $\mathscr{H}$ be a monotonically decreasing sequence of positive numbers that tends to zero, and let $(g_h)_{h \in \mathscr{H}}$ be a family of real-valued functions, defined on $\mathbb{R}^d$, such that*

$$|g_h(x)| \leqslant c_0(1 + \|x\|)^\tau, \qquad x \in \mathbb{R}^d, \quad h \in \mathscr{H}, \tag{2.24}$$

where c_0 and τ are positive constants. If τ satisfies the inequality

$$l - \tau > d, \tag{2.25}$$

then the condition

$$\sum_{j \in \mathbb{Z}^d \backslash h^{-1} K_2} |g_h(hj)| \, |\chi(h^{-1}x - j)| = \mathcal{O}(h^{l-d}), \tag{2.26}$$

holds uniformly for $x \in K_1$, as $h \to 0$.

Proof. Let $\Delta = \max\{\|x\|: x \in K_1\}$ and let δ be the distance between K_1 and the boundary of K_2. By (2.24), for any $j \in \mathbb{Z}^d \backslash h^{-1} K_2$ and any $x \in K_1$, we deduce

$$\begin{aligned} |g_h(hj)| &\leqslant c_0(1 + \|h_j\|)^\tau \\ &\leqslant c_0(1 + \|x\| + \|hj - x\|)^\tau \\ &\leqslant c_0 \, \|hj - x\|^\tau \left(1 + \frac{1 + \Delta}{\delta}\right)^\tau. \end{aligned} \tag{2.27}$$

Then we can use (1.7) and (2.27) to obtain

$$\sum_{j \in \mathbb{Z}^d \setminus h^{-1}K_2} |g_h(hj)| \, |\chi(h^{-1}x - j)| \leqslant c_0 h^\tau \left(1 + \frac{1+\Delta}{\delta}\right)^\tau \sum_{j \in \mathbb{Z}^d \setminus h^{-1}K_2} \frac{1}{\|h^{-1}x - j\|^{l-\tau}}. \tag{2.28}$$

Moreover, there exists a constant $c_1 > 0$ such that

$$\begin{aligned}\sum_{j \in \mathbb{Z}^d \setminus h^{-1}K_2} \frac{1}{\|h^{-1}x - j\|^{l-\tau}} &\leqslant c_1 \int_{\|t\| \geqslant h^{-1}\delta} \|t\|^{-l+\tau} \, dt \\ &= c_1 \sigma_d \int_{r = h^{-1}\delta}^{\infty} r^{-l+\tau+d-1} \, dr \\ &= \frac{c_1 \sigma_d}{l - \tau - d} \left(\frac{h}{\delta}\right)^{l-\tau-d},\end{aligned} \tag{2.29}$$

where σ_d is the surface area of the unit sphere in $\mathbb{R}^d$. The assumption (2.25) ensures that the integrals are finite. Now (2.28) and (2.29) imply (2.26), as required. ▮

The hypotheses of Proposition 2 are satisfied if we let $K_1 = K$ (the compact set from Theorem 1), $K_2 = D$, $g_h = f^* - s_h$ for all sufficiently small h, and we let χ be the cardinal function (1.5). Indeed, in this case (1.7) holds for $l = 2d + \gamma$, while (2.24) is satisfied by $c_0 = c_f + \|f^*\|_{L^\infty(\mathbb{R}^d)}$, where τ is given by Proposition 1. For these values, (2.25) is true when $\gamma > 0$, $\gamma \notin 2\mathbb{N}$, since the constant σ of (2.22) is zero in this case. Further, (2.25) is also satisfied when $\gamma \in 2\mathbb{N}_+$ by choosing $\sigma \in (0, 1)$ for this situation. Therefore (1.11), (1.2), Proposition 2 and $l - d = d + \gamma$ provide

$$\begin{aligned}|I_h f^*(x) - s_h(x)| &= \left| \sum_{j \in \mathbb{Z}^d \setminus h^{-1}D} (f^* - s_h)(hj) \, \chi(h^{-1}x - j) \right| \\ &\leqslant \sum_{j \in \mathbb{Z}^d \setminus h^{-1}D} |(f^* - s_h)(hj)| \, |\chi(h^{-1}x - j)| \\ &= \mathcal{O}(h^{d+\gamma}), \qquad \text{as} \quad h \to 0,\end{aligned} \tag{2.30}$$

uniformly for $x \in K$. We now combine (1.4), (1.10) and (2.30) to obtain the desired conclusion (1.13). This finishes the proof of Theorem 1. ▮

Remark. When the decay of χ is exponential, Proposition 2 implies that the sum (2.26) tends to zero faster than any power of h. This property allows the proof of Theorem 2 that is given in the next section. However, the value $l = 2d + \gamma$ in (1.7) is sufficient to establish Theorem 1 for all the choices of ϕ considered in this paper.

3. PROOF OF THEOREM 2

We use the method of Jackson [6, Proof of Theorem 5–10] and Buhmann [3, Proof of Theorem 5–13], originally developed for the cases of quasi-interpolation and interpolation, respectively, on infinite regular grids. We note that γ is an integer in Theorem 2, which implies $\kappa = d + \gamma - 1$.

Suppose that the convergence rate

$$\|f - s_h\|_{L^\infty(K)} = o(h^{d+\gamma}), \qquad h \to 0 \tag{3.1}$$

holds for all functions $f \in \mathrm{Lip}(\kappa + 3, D)$ and all compact sets $K \subset \mathrm{int}(D)$. Then we will deduce below the property

$$Ip \equiv p \qquad \text{for all} \quad p \in \Pi_{\kappa+1}(\mathbb{R}^d), \tag{3.2}$$

where I is the interpolation operator (1.6). Since κ is the maximal degree for reproduction of polynomials on the multi-integer grid [4], however, Eq. (3.2) provides a contradiction that establishes Theorem 2.

LEMMA 1. *Let p be a homogeneous polynomial of total degree $\kappa + 1$ on $\mathbb{R}^d$ and let K be a compact subset of the interior of D that has $\frac{1}{2}u$ as an interior point, where $u = (1, 1, ..., 1)^T \in \mathbb{Z}^d$. If assumption (3.1) is true, then the bound*

$$|p(x - \tfrac{1}{2}u)| - \sum_{j \in \mathbb{Z}^d} p(hj - \tfrac{1}{2}u)\, \chi(h^{-1}x - j) = o(h^{\kappa+1}), \qquad h \to 0, \tag{3.3}$$

holds uniformly for $x \in K$.

Proof. Let $g(x) = p(x - \frac{1}{2}u)$ for all $x \in \mathbb{R}^d$. Given a compact $K \subset \mathrm{int}(D)$, we let M be another compact such that $K \subset \mathrm{int}(M)$ and $M \subset \mathrm{int}(D)$. Let $\rho \in C_0^\infty(\mathbb{R}^d)$ be a cut-off function with the properties: $0 \leqslant \rho(x) \leqslant 1$ for $x \in \mathbb{R}^d$, $\rho(x) = 1$ for $x \in M$, and $\mathrm{supp}(\rho) \subset \mathrm{int}(D)$. Hence the function $f\colon \mathbb{R}^d \to \mathbb{R}$, $f(x) = \rho(x)\, g(x)$ is infinitely differentiable, with $\mathrm{supp}(f) \subset \mathrm{int}(D)$. In particular, f satisfies the properties required for the extension f of $f|_D$ to $\mathbb{R}^d$ that is constructed at the beginning of Section 2, so we can let $f^* = f$ on $\mathbb{R}^d$.

We replace K by M in assumption (3.1) for the above f. Then the properties $\rho|_M \equiv 1$ and $d+\gamma=\kappa+1$ imply

$$|g(x)-s_h(x)|=o(h^{\kappa+1}), \qquad h\to 0, \tag{3.4}$$

uniformly for $x\in M$, where s_h is the RBF interpolant to $f|_D$ on the finite grid $\mathscr{V}_h$. We let $I_h f^*$ and $I_h g$ be the RBF interpolants to $f^*=f$ and g on the infinite grid $h\mathbb{Z}^d$, respectively, as given by (1.8). The exponential decay of the cardinal function χ ensures that $I_h g$ is well defined and that $l=d+\kappa+2$ is allowed in (1.7). For this value of l, we apply Proposition 2 twice.

First, we let $K_1=M$, $K_2=D$ and $g_h=s_h$, $h>0$ in Proposition 2. Then (2.24) holds with $c_0=c_f$, the value of τ being given by Proposition 1. Further, (2.25) is also satisfied in this case, since the constant σ of (2.22) is zero when $\gamma\in 2\mathbb{N}+1$, and we choose $\sigma\in(0,1)$ when $\gamma\in 2\mathbb{N}_+$. Therefore (1.11), the compact support of f^* and Proposition 2 provide

$$\begin{aligned}|s_h(x)-I_h f^*(x)| &= |I_h(s_h-f^*)(x)|\\ &= \Bigg|\sum_{j\in\mathbb{Z}^d\setminus h^{-1}D} s_h(hj)\,\chi(h^{-1}x-j)\Bigg|\\ &\leqslant \sum_{j\in\mathbb{Z}^d\setminus h^{-1}D} |s_h(hj)|\,|\chi(h^{-1}x-j)|\\ &= \mathcal{O}(h^{\kappa+2}), \qquad \text{as} \quad h\to 0,\end{aligned} \tag{3.5}$$

uniformly for $x\in M$.

Second, let $K_1=K$, $K_2=M$ and $g_h=g$, $h>0$ (i.e., the family $(g_h)_{h>0}$ reduces to the single element g) in Proposition 2. Since $g\in\Pi_{\kappa+1}(\mathbb{R}^d)$, it follows that (2.24) holds with $\tau=\kappa+1$, and condition (2.25) becomes $l-\tau=d+1>d$. Hence the properties $\rho|_M\equiv 1$, $0\leqslant\rho\leqslant 1$ and another application of Proposition 2 imply

$$\begin{aligned}|I_h f^*(x)-I_h g(x)| &= |I_h(f^*-g)(x)|\\ &\leqslant \Bigg|\sum_{j\in\mathbb{Z}^d\setminus h^{-1}M} [1-\rho(hj)]\; g(hj)\,\chi(h^{-1}x-j)\Bigg|\\ &\leqslant \sum_{j\in\mathbb{Z}^d\setminus h^{-1}M} |g(hj)|\,|\chi(h^{-1}x-j)|\\ &= \mathcal{O}(h^{\kappa+2}), \qquad \text{as} \quad h\to 0,\end{aligned} \tag{3.6}$$

uniformly for $x\in K$.

Therefore (3.4)–(3.6) prove (3.3), as required. ∎

We next use the following change of variables. We let $y = x - \frac{1}{2}u$, we work with $N = 2\nu$ for increasing positive integers ν, and we recall $h = N^{-1}$. Then, if $j \in \mathbb{Z}^d$ and $k = j - \nu u$, we have $hj - \frac{1}{2}u = hk$ and $h^{-1}y - k = h^{-1}x - j$. Hence (3.3) is equivalent to

$$\left| p(y) - \sum_{k \in \mathbb{Z}^d} p(hk)\,\chi(h^{-1}y - k) \right| = o(h^{\kappa+1}), \qquad h \to 0, \tag{3.7}$$

uniformly for $y \in K - \frac{1}{2}u$.

Now the proof of (3.2) depends on the remark that $K - \frac{1}{2}u$ contains a neighborhood of the origin in $\mathbb{R}^d$, and on a classical homogeneity argument. Specifically, if $z \in \mathbb{R}^d$ is fixed, then $hz \in K - \frac{1}{2}u$ for any sufficiently small h, so (3.7) implies

$$\left| p(hz) - \sum_{k \in \mathbb{Z}^d} p(hk)\,\chi(z - k) \right| = o(h^{\kappa+1}), \qquad h \to 0. \tag{3.8}$$

Therefore the homogeneity of p yields

$$\left| p(z) - \sum_{k \in \mathbb{Z}^d} p(k)\,\chi(z - k) \right| = o(1), \qquad h \to 0, \tag{3.9}$$

which proves $Ip(z) = p(z)$, because the left-hand side of (3.9) is independent of h. Since z is arbitrary, (3.2) follows, which completes the proof of Theorem 2.

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REFERENCES

1. K. E. Atkinson, On the order of convergence of natural cubic spline interpolation, *SIAM J. Numer. Anal.* **5** (1968), 89–101.
2. C. de Boor and A. Ron, Fourier analysis of the approximation power of principal shift-invariant spaces, *Constr. Approx.* **8** (1992), 427–462.
3. M. D. Buhmann, "Multivariable Interpolation Using Radial Basis Functions," Ph.D. Dissertation, University of Cambridge, 1989.
4. M. D. Buhmann, Multivariate cardinal interpolation with radial-basis functions, *Constr. Approx.* **6** (1990), 225–255.
5. J. Duchon, Sur l'erreur d'interpolation des fonctions de plusieurs variables par les D^m-splines, *RAIRO An. Num.* **12**, No. 4 (1978), 325–334.

6. I. R. H. Jackson, "Radial Basis Function Methods for Multivariable Approximation," Ph.D. Dissertation, University of Cambridge, 1988.
7. I. R. H. Jackson, Radial basis functions: A survey and new results, *in* "The Mathematics of Surfaces, III" (D. C. Handscomb, Ed.), pp. 115–133, Oxford Univ. Press, Oxford, 1989.
8. M. Johnson, A bound on the approximation order of surface splines, *Constr. Approx.* **14** (1998), 429–438.
9. M. Johnson, An improved order of approximation for thin-plate spline interpolation in the unit disc, preprint, Kuwait University, 1998.
10. W. R. Madych and S. A. Nelson, Polyharmonic cardinal splines, *J. Approx. Theory* **60** (1990), 141–156.
11. O. V. Matveev, On a method for interpolating functions on chaotic nets, *Math. Notes* **62**, No. 3 (1997), 339–349; *Mat. Zametki* **62**, No. 3 (1997), 404–417.
12. C. A. Micchelli, Interpolation of scattered data: Distance matrices and conditionally positive definite functions, *Constr. Approx.* **2** (1986), 11–22.
13. M. J. D. Powell, The theory of radial basis function approximation in 1990, *in* "Advances in Numerical Analysis, Vol. II, Wavelets, Subdivision Algorithms, and Radial Basis Functions" (W. A. Light, Ed.), pp. 105–210, Clarendon, Oxford, 1992.
14. M. J. D. Powell, The uniform convergence of thin plate spline interpolation in two dimensions, *Numer. Math.* **68** (1994), 107–128.
15. M. J. D. Powell, A review of methods for multivariable interpolation at scattered data points, *in* "The State of the Art in Numerical Analysis" (I. S. Duff and G. A. Watson, Eds.), pp. 283–309, Clarendon, Oxford, 1997.
16. R. Schaback, Comparison of radial basis function interpolants, *in* "From CAGD to Wavelets" (K. Jetter and F. Utreras, Eds.), pp. 293–305, World Scientific, Singapore, 1993.
17. R. Schaback, Approximation by radial basis functions with finitely many centers, *Constr. Approx.* **12** (1996), 331–340.
18. R. Schaback, Improved error bounds for scattered data interpolation by radial basis functions, *Math. Comp.* **68** (1999), 201–216.
20. E. M. Stein, "Singular Integrals and Differentiability Properties of Functions," Princeton Univ. Press, Princeton, 1970.
21. Z.-M. Wu and R. Schaback, Local error estimates for radial basis function interpolation of scattered data, *IMA J. Numer. Anal.* **13** (1993), 13–27.